Two Modified Algorithms with Order of Convergence Five for Finding a Simple Root Of Nonlinear Equations

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Abstract—In this paper, we present two modified algorithms with order of convergence five for finding a simple root \( x^* \) of nonlinear scalar equation \( f(x)=0 \) in \( \mathbb{R} \). It is free from second derivatives. Both of them require two evaluations of the functions and two evaluations of derivatives at each iteration. Therefore the efficiency index of the presented methods is 1.4953 which is better than that of classical Newton’s method 1.4142. Some numerical results illustrate the effectiveness and performance of the presented methods.

Keywords—Nonlinear equations; Newton’s method; Order of convergence; Iterative method; Efficiency index

I. INTRODUCTION

One of the most important problems in scientific and engineering applications is to solve nonlinear equations. This paper is concerned with iterative methods to find a simple root \( x^* \) of nonlinear scalar equation \( f(x)=0 \) in \( \mathbb{R} \), where \( f : D \subseteq \mathbb{R} \rightarrow \mathbb{R} \) for an open interval \( D \) is a scalar function and it is sufficiently smooth in a neighborhood of \( x^* \).

There are many iterative methods such as Newton’s method and its variants (see references (see [1]-[10] for more details). Among these methods, classical Newton's method is the best known and probably the most used algorithm for solving nonlinear equations by using the following iterative scheme

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}
\]

which is quadratically convergent in a neighborhood of \( x^* \) [1].

In past decades, much attention has been paid to develop iterative methods for solving nonlinear equations, and many iterative methods have been developed. In this paper, we present two modified three-step iterative methods for solving nonlinear equations. Analysis of convergence shows that the new methods have fifth-order convergence. Both of them are free from second derivatives. They require three evaluations of the functions and two evaluations of derivatives at each iteration. The efficiency index of the presented methods is better than that of classical Newton’s method.

II. THE MODIFIED METHODS AND ITS CONVERGENCE

Let us consider the following iterative algorithm.

Algorithm 1. For given \( x_0 \), we consider the following iteration scheme for solving nonlinear equation \( f(x)=0 \) in \( \mathbb{R} \)

\[
y_n = x_n - \frac{2f(x_n)}{3f'(x_n)},
\]

\((2)\)
\[ z_n = x_n - \frac{f'(x_n) + 3f'(y_n)}{-2f'(x_n) + 6f'(y_n)} \frac{f(x_n)}{f'(x_n)}, \]  
\[ x_{n+1} = z_n + \frac{2f'(x_n)^2}{f'(x_n)^2 - 4f'(x_n)f'(y_n) + f'(y_n)^2} \frac{f(z_n)}{f'(x_n)}. \]

For Algorithm 1, we have the following convergence result.

**THEOREM 1.** Assume that the function \( f : D \subseteq \mathbb{R} \rightarrow \mathbb{R} \) has a single root \( x^* \in D \), where \( D \) is an open interval. If \( f(x) \) has first, second and third derivatives in the interval \( D \), then Algorithm 1 defined by (2)-(4) is fifth-order convergent in a neighborhood of \( x^* \) and it satisfies error equation

\[ e_{n+1} = 80c_2(c_2^3 - c_2c_3 + \frac{1}{9}c_4)e_n^5 + O(e_n^6) \]

where

\[ e_n = x_n - x^*, \quad c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}, \quad k = 1, 2, \ldots. \]

**Proof.** Let \( x^* \) be the simple root of \( f(x) \), \( c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}, \quad k = 1, 2, \ldots \), and \( e_n = x_n - x^* \). Consider the iteration function \( F(x) \) defined by

\[ F(x) = z(x) + \frac{2f'(x)^2}{f'(x)^2 - 4f'(x)f'(y(x)) + f'(y(x))^2} \frac{f(z(x))}{f'(x)} \]

where

\[ z(x) = y(x) - \frac{f'(x) + 3f'(y(x))}{-2f'(x) + 6f'(y(x))} \frac{f(x)}{f'(x)}, \quad y(x) = x - \frac{2f(x)}{3f'(x)}. \]

By some computations using Maple we can obtain

\[ F(x^*) = x^*, \quad F^{(i)}(x^*) = 0, \quad i = 1, 2, 3, 4, \]

\[ F^{(5)}(x) = \frac{5f^*(x^*)}{27f'(x^*)^4} \left[-18f^*(x^*)f^{(3)}(x^*)f'(x^*)f'(y(x))^2 + 27f^*(x^*)f'(x^*)^2 \right]. \]

Furthermore, from the Taylor expansion of \( F(x_n) \) at \( x^* \), we get

\[ x_{n+1} = F(x_n) = F(x^*) + \sum_{k=1}^{5} \frac{F^{(k)}(x^*)}{k!} (x_n - x^*)^k + O((x_n - x^*)^6). \]

Substituting (8) into (9) yields

\[ x_{n+1} = x^* + e_{n+1} = x^* + 80c_2(c_2^3 - c_2c_3 + \frac{1}{9}c_4)e_n^5 + O(e_n^6). \]
Therefore, we have
\[ e_{n+1} = 80c_2(c_2^3 - c_2c_3 + \frac{1}{9}c_4)e_n^5 + O(e_n^6), \]
which means the order of convergence of the Algorithm 1 is five. The proof is completed.

Similarly, we give the following fifth-order convergent iterative method and its convergence analysis. We omit the proof here for simplicity.

**Algorithm 2.** For given \( x_0 \), we consider the following iteration method
\[
\begin{align*}
y_n &= x_n - \frac{2f(x_n)}{3f'(x_n)}, \\
z_n &= x_n - \frac{f'(x_n) + 3f'(y_n)}{2f'(x_n) + 6f'(y_n)} f(x_n), \\
x_{n+1} &= z_n - \frac{2f'(y_n)^2}{f'(x_n)^2 - 4f'(x_n)f'(y_n) + 5f'(y_n)^2} f(z_n). 
\end{align*}
\]

For Algorithm 2, we have the following convergence result.

**THEOREM 2.** Assume that the function \( f:D\subseteq R \rightarrow R \) has a single root \( x^* \in D \), where \( D \) is an open interval. If \( f(x) \) has first, second and third derivatives in the interval \( D \), then Algorithm 2 defined by (10)-(12) is fifth-order convergent in a neighborhood of \( x^* \) and its error equation is
\[ e_{n+1} = 80c_2(c_2^3 - c_2c_3 + \frac{1}{9}c_4)e_n^5 + O(e_n^6) \]
where \( e_n \) and \( c_k \) (\( k = 1, 2, \cdots \)) are defined by (6).

Now, we consider efficiency index defined as \( \frac{p}{\omega} \), where \( p \) is the order of the method and \( \omega \) is the number of function evaluations per iteration required by the method. It is not hard to see that the efficiency index of Algorithm 1 and Algorithm 2 is 1.4953 which is better than that of classical Newton’s method (NM) 1.4142.

**III. CONCLUSIONS**
In this paper, we present and analyze two modified fifth-order convergent iterative methods for solving nonlinear equations. They are free from second derivatives, and both of them require two evaluations of the functions and two evaluations of derivatives in each step. The fifth-order methods proposed in this paper are more efficient than Newton's method.

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